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Rare decay modes of quarter BPS dyons

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Abstract: The degeneracy of quarter BPS dyons in N=4 supersymmetric string theories is known to jump across walls of marginal stability on which a quarter BPS dyon can decay into a pair of half BPS dyons. We show that as long as the electric and magnetic charges of the original dyon are primitive elements of the charge lattice, the subspaces of the moduli space on which a quarter BPS dyon becomes marginally unstable against decay into a pair of quarter BPS dyons or a half BPS dyon and a quarter BPS dyon are of codimension two or more. As a result any pair of generic points in the moduli space can be connected by a path avoiding these subspaces and there is no jump in the spectrum associated with these subspaces.

Keywords: Superstrings and Heterotic Strings, Black Holes in String Theory.

We now have a good understanding of the exact spectrum of a class of quarter BPS dyons in $\mathcal{N}=4$ supersymmetric string theories [1−16]. It is also known that as we cross various walls of marginal stability associated with the possible decay of the dyon into a pair of half BPS states, the degeneracy changes by a certain amount that is exactly computable [12, 13]. Furthermore in the gravity description this jump can be accounted for by the (dis)appearance of two centered small black holes [17−22] as in the asymptotic moduli space we cross walls of marginal stability [13, 15, 16]. This raises the question: why aren't there similar effects associated with the decay of a quarter BPS dyon into a pair of quarter BPS dyons, or into a quarter BPS dyon and a half BPS dyon? In this note we shall show that such decays take place on subspaces of codimension higher than one as long as electric and magnetic charge vectors of the original dyon are primitive elements of the charge lattice. Hence we can move from any generic point in the moduli space to another generic point in the moduli space without ever passing through these subspaces, and there is no effect of the type discussed in [12, 13, 15, 16] associated with these decays.[1]

We denote by r the total number of U(1) gauge fields in the model, by $\vec{Q}$ and $\vec{P}$ the r dimensional electric and the magnetic charge vectors, by $\tau = a + iS$ the axion-dilaton moduli field parametrizing the upper half plane and by M the $r\times r$ matrix valued scalar field satisfying

$$MLM^T = L, \qquad M^T = M, \qquad L = \begin{pmatrix} I_6 & \\ & -I_{r-6} \end{pmatrix} \tag{1}$$

where I_k denotes $k\times k$ identity matrix. We shall use the subscript $_\infty$ to denote the asymptotic values of various scalar fields. Let us now introduce the SO$(6, r-6)$ matrix Ω_∞ via the relations[2]

$$M_\infty = \Omega_\infty \Omega_\infty^T, \qquad \Omega_\infty L \Omega_\infty^T = L\,, \tag{2}$$

and define

$$Q_R = \frac{1}{2}(I_r + L)\Omega_\infty^T\, Q, \qquad P_R = \frac{1}{2}(I_r + L)\Omega_\infty^T\, P\,. \tag{3}$$

The vectors $\vec{Q}_R$ and $\vec{P}_R$ lie in the six dimentional subspace spanned by the eigenvectors of L with eigenvalue 1. In terms of $\vec{Q}_R$ and $\vec{P}_R$ the BPS mass formula of [25, 26] takes the form [12],

$$m(\vec{Q}, \vec{P}) = \sqrt{2}\, f(\vec{Q}_R, \vec{P}_R; a_\infty, S_\infty)\,, \tag{4}$$

where

$$f(\vec{Q}_R, \vec{P}_R; a_\infty, S_\infty) = \sqrt{\frac{1}{S_\infty}(\vec{Q}_R - a_\infty \vec{P}_R)^2 + S_\infty \vec{P}_R^2 + 2\left[\vec{Q}_R^2 \vec{P}_R^2 - (\vec{Q}_R\cdot\vec{P}_R)^2\right]^{1/2}}\,. \tag{5}$$

The inner products of $\vec{Q}_R$ and $\vec{P}_R$ are calculated with the identity matrix.

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[1] Different approaches to this problem have been advocated in [23, 24].

[2] Since (2) is invariant under a right multiplication of Ω_∞ by an SO(6) × SO$(r-6)$ matrix that preserves both the identity matrix and L, (2) does not fix Ω_∞ completely in terms of M_∞. This problem may be avoided by choosing a suitable 'gauge condition' on Ω_∞ so that there is one to one correspondence between M_∞ and Ω_∞.

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Let us now consider a possible marginal decay $(\vec{Q},\vec{P}) \to (\vec{Q}_1,\vec{P}_1) + (\vec{Q}-\vec{Q}_1,\vec{P}-\vec{P}_1)$. This requires adjusting the moduli such that

$$f(\vec{Q}_R,\vec{P}_R;a_\infty,S_\infty) = f(\vec{Q}_{1R},\vec{P}_{1R};a_\infty,S_\infty) + f(\vec{Q}_R-\vec{Q}_{1R},\vec{P}_R-\vec{P}_{1R};a_\infty,S_\infty)\,. \tag{6}$$

For fixed $\vec{Q}$, $\vec{P}$ and M_∞, $\vec{Q}_R$ and $\vec{P}_R$ span a two dimensional subspace of the six dimensional space on which L has eigenvalue $+1$. Let us denote by $\vec{Q}_{1R\|}$ and $\vec{P}_{1R\|}$ the projection of $\vec{Q}_{1R}$ and $\vec{P}_{1R}$ along this two dimensional subspace. Then we have the following inequalities:

$$\begin{aligned} f(\vec{Q}_R,\vec{P}_R;a_\infty,S_\infty) &\le f(\vec{Q}_{1R\|},\vec{P}_{1R\|};a_\infty,S_\infty) \\ &\qquad + f(\vec{Q}_R-\vec{Q}_{1R\|},\vec{P}_R-\vec{P}_{1R\|};a_\infty,S_\infty)\,, && (7)\\ f(\vec{Q}_{1R\|},\vec{P}_{1R\|};a_\infty,S_\infty) &\le f(\vec{Q}_{1R},\vec{P}_{1R};a_\infty,S_\infty)\,, && (8)\\ f(\vec{Q}_R-\vec{Q}_{1R\|},\vec{P}_R-\vec{P}_{1R\|};a_\infty,S_\infty) &\le f(\vec{Q}_R-\vec{Q}_{1R},\vec{P}_R-\vec{P}_{1R};a_\infty,S_\infty)\,. && (9)\end{aligned}$$

The inequality (7) is proved by defining

$$\vec{a} = \frac{\vec{Q}_R - a_\infty\vec{P}_R}{\sqrt{S_\infty}}, \qquad \vec{b} = \vec{P}_R\sqrt{S_\infty}\,, \qquad \vec{a}_1 = \frac{\vec{Q}_{1R\|} - a_\infty\vec{P}_{1R\|}}{\sqrt{S_\infty}}, \qquad \vec{b}_1 = \vec{P}_{1R\|}\sqrt{S_\infty}\,, \tag{10}$$

and using the inequality:

$$\sqrt{\vec{a}^2+\vec{b}^2+2|\vec{a}\times\vec{b}|} \le \sqrt{\vec{a}_1^2+\vec{b}_1^2+2|\vec{a}_1\times\vec{b}_1|} + \sqrt{(\vec{a}-\vec{a}_1)^2+(\vec{b}-\vec{b}_1)^2+2|(\vec{a}-\vec{a}_1)\times(\vec{b}-\vec{b}_1)|} \tag{11}$$

for any set of vectors $\vec{a}$, $\vec{b}$, $\vec{a}_1$, $\vec{b}_1$ lying in a two dimensional plane. (11) can be easily proven with the help of triangle inequality if we note that $\sqrt{\vec{a}^2+\vec{b}^2+2|\vec{a}\times\vec{b}|}$ can be interpreted as $|\vec{a}+\epsilon\vec{b}|$ where ϵ is the $\pi/2$ rotation matrix in the plane of $\vec{a}$ and $\vec{b}$, with the sign of ϵ chosen such that $a^T\epsilon b > 0$. Requiring the inequality (11) to be saturated gives one equation and several inequalities among the components of $\vec{a}$, $\vec{b}$, $\vec{a}_1$ and $\vec{b}_1$:

$$\vec{a}_1 + \epsilon\vec{b}_1 = \lambda(\vec{a}+\epsilon\vec{b}) \quad \text{with} \quad 0\le\lambda\le 1\,, \qquad a_1^T\epsilon b_1 \ge 0, \qquad (a-a_1)^T\epsilon(b-b_1)\ge 0\,. \tag{12}$$

Using (10) we can translate these conditions into one constraint equation and some inequalities involving the variables $(a_\infty, S_\infty, M_\infty, \vec{Q}, \vec{P})$.

The inequality (8) follows from the observations that

$$\begin{aligned} |\vec{Q}_{1R\|} - \bar\tau_\infty\vec{P}_{1R\|}|^2 &\le |\vec{Q}_{1R} - \bar\tau_\infty\vec{P}_{1R}|^2, \\ \sqrt{\vec{Q}_{1R\|}^2\vec{P}_{1R\|}^2 - (\vec{Q}_{1R\|}\cdot\vec{P}_{1R\|})^2} &\le \sqrt{\vec{Q}_{1R}^2\vec{P}_{1R}^2 - (\vec{Q}_{1R}\cdot\vec{P}_{1R})^2}\,. \end{aligned} \tag{13}$$

The first of these is obvious since the (complex) vector on the left hand side is a projection of the vector on the right hand side along the plane spanned by $\vec{Q}_R$ and $\vec{P}_R$. The second one follows from the fact that the right hand side of the inequality represents the area of a triangle formed by the vectors $\vec{Q}_{1R}$ and $\vec{P}_{1R}$ and the left hand side represents the area of the projection of this triangle in the plane spanned by $\vec{Q}_R$ and $\vec{P}_R$. Both inequalities are saturated when $\vec{Q}_{1R}$ and $\vec{P}_{1R}$ lie in the plane spanned by $\vec{Q}_R$ and $\vec{P}_R$. The inequality (9) can be proved in an identical manner and is also saturated when $\vec{Q}_{1R}$ and $\vec{P}_{1R}$ lie in the plane spanned by $\vec{Q}_R$ and $\vec{P}_R$. This requires adjusting Ω_∞ or equivalently M_∞ appropriately.

Now in order to satisfy the condition for marginal stability (6) we must saturate all the three inequalities (7)-(9). This would require adjusting moduli M_∞ to make $(\vec{Q}_{1R}, \vec{P}_{1R})$ lie in the plane of $(\vec{Q}_R, \vec{P}_R)$, and additional adjustment of (a_∞, S_∞) to saturate the inequality (7). Thus we have a surface of codimension two or more, and we can go from any generic point in the moduli space to another generic point in the moduli space without ever encountering this subspace of marginal stability. This shows that there is no discontinuous change in the spectrum associated with these subspaces.

It is instructive to compare this with the condition for marginal stability of half BPS dyons in the $\mathcal{N}=2$ supersymmetric S-T-U model. In that case the BPS mass formula is identical to the one given in (4), (5), but M and L are 4×4 matrices and L has two eigenvalues $+1$ and two eigenvalues -1. As a result the vectors $\vec{Q}_R$, $\vec{P}_R$, $\vec{Q}_{1R}$, $\vec{P}_{1R}$ all lie inside a two dimensional subspace spanned by the eigenvectors of L with eigenvalue $+1$, and the inequalities (8), (9) are automatically saturated. Thus we only need to saturate the inequality (7). This gives one condition on the asymptotic moduli, producing a codimension one surface.

There is a special case where our argument fails for the $\mathcal{N}=4$ supersymmetric theory. If the full r dimensional charge vectors $\vec{Q}_1$ and $\vec{P}_1$ happen to lie in the plane spanned by $\vec{Q}$ and $\vec{P}$ then $\vec{Q}_{1R}$ and $\vec{P}_{1R}$ automatically lie in the plane of $\vec{Q}_R$ and $\vec{P}_R$ and we do not get any condition on the moduli M_∞ from (8), (9). This would require $\vec{Q}_1$ and $\vec{P}_1$ to be of the form:

$$\vec{Q}_1 = \alpha\vec{Q} + \beta\vec{P}, \qquad \vec{P}_1 = \gamma\vec{Q} + \delta\vec{P}\,. \tag{14}$$

If we take $\vec{Q}$ and $\vec{P}$ to be primitive then charge quantization would require α, β, γ, δ to be integers. Furthermore in the $\mathbb{Z}_N$ orbifold models γ must be integer multiples of N since in one particular direction along the charge lattice Q is quantized in units of $1/N$ while P is quantized in integer units [5, 8]. (14) now implies that

$$\vec{Q}_{1R} = \alpha\vec{Q}_R + \beta\vec{P}_R, \qquad \vec{P}_{1R} = \gamma\vec{Q}_R + \delta\vec{P}_R\,. \tag{15}$$

We can substitute these into (7) and use the discussion below (11) to determine under what condition the inequality might be saturated. We shall only consider the case when $\vec{Q}_R$ and $\vec{P}_R$ are not parallel, i.e. $\vec{Q}_R^2\vec{P}_R^2 - (\vec{Q}_R\cdot\vec{P}_R)^2 \neq 0$, since for generic $\vec{Q}$ and $\vec{P}$ aligning $\vec{Q}_R$ and $\vec{P}_R$ will impose more than one condition on Ω_∞ and will produce a surface of codimension higher than one. In this case the $a_1^T\epsilon b_1 \geq 0$, $(a-a_1)^T\epsilon(b-b_1)\geq 0$ conditions give

$$\alpha\delta - \beta\gamma \geq 0\,, \qquad (1-\alpha)(1-\delta) - \beta\gamma \geq 0\,. \tag{16}$$

On the other hand the $(\vec{a}_1 + \epsilon\vec{b}_1) = \lambda(\vec{a}+\epsilon\vec{b})$ condition gives

$$(\alpha-\lambda-a_\infty\gamma)\sqrt{\vec{Q}_R^2\vec{P}_R^2 - (\vec{Q}_R\cdot\vec{P}_R)^2} + S_\infty\left(\gamma\vec{Q}_R\cdot\vec{P}_R + (\delta-\lambda)\vec{P}_R^2\right) = 0\,, \tag{17}$$

$$(\beta-(\delta-\lambda)a_\infty)\sqrt{\vec{Q}_R^2\vec{P}_R^2 - (\vec{Q}_R\cdot\vec{P}_R)^2} - S_\infty\left(\gamma\vec{Q}_R^2 + (\delta-\lambda)\vec{Q}_R\cdot\vec{P}_R\right) = 0\,, \tag{18}$$

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where λ is an arbitrary parameter with

$$0 \le \lambda \le 1\,. \tag{19}$$

We can solve for a_∞ using (17) and substitute into (18) to get

$$\left(\beta\gamma - \alpha\delta + \lambda(\alpha+\delta) - \lambda^2\right)\sqrt{\vec{Q}_R^2\vec{P}_R^2 - (\vec{Q}_R\cdot\vec{P}_R)^2} - S_\infty\left\{(\lambda-\delta)\vec{P}_R - \gamma\vec{Q}_R\right\}^2 = 0\,. \tag{20}$$

If $\gamma = 0$ then (18) contains additional information beyond what can be obtained from (20) and (17).

We focus on the constraint (20). Since the second term proportional to S_∞ is negative semi-definite, if we can show that the first term is also negative definite then we would have shown that the equation has no solution. For this analysis we shall make use of (16). Let us consider the following cases separately:

- First consider the case when both the inequalities in (16) are saturated:

$$\alpha\delta - \beta\gamma = 0\,, \qquad (1-\alpha)(1-\delta) - \beta\gamma = 0\,. \tag{21}$$

  This correponds to the case when in each of the decay products the electric and the magnetic charge vectors are parallel. Hence both the decay products are half BPS. (21) gives $\alpha + \delta = 1$. Using this we can express the first term on the left hand side of (20) as

$$\left(\lambda - \lambda^2\right)\sqrt{\vec{Q}_R^2\vec{P}_R^2 - (\vec{Q}_R\cdot\vec{P}_R)^2}\,. \tag{22}$$

  Since this is positive semi-definite in the range (19) this can cancel the second term in (20) on a codimension 1 subspace in the (λ, S_∞) space. Using (17) or (18) we can convert this to a codimension 1 subspace in the (a_∞, S_∞) space, reproducing the marginal stability walls studied in [12].

- Now consider the case where at least one of the decay products is quarter BPS. Without any loss of generality we can take this to be the state carrying charges $(\alpha\vec{Q}+\beta\vec{P}, \gamma\vec{Q}+\delta\vec{P})$. In this case the first inequality in (16) will be a strict inequality. Combining this with the information that $\alpha, \beta, \gamma, \delta$ are integers we get

$$\alpha\delta - \beta\gamma \ge 1\,, \qquad (1-\alpha)(1-\delta) - \beta\gamma \ge 0\,. \tag{23}$$

  Our goal is to use these results to analyze the first term on the left hand side of (20). Due to (23),

$$\left(\beta\gamma - \alpha\delta + \lambda(\alpha+\delta) - \lambda^2\right) \tag{24}$$

  is negative or zero at $\lambda = 0, 1$. Thus in order for it to be positive in some range of value between $\lambda = 0$ and $\lambda = 1$, it must have a maximum in this range, and its value at the maximum must be positive. Now (24) has a maximum at

$$\lambda = \frac{\alpha+\delta}{2}\,, \tag{25}$$

where it takes the value

$$\frac{1}{4}(\alpha+\delta)^2 - (\alpha\delta - \beta\gamma)\,. \tag{26}$$

Eq. (25) shows that in order that the maximum lies in the range $(0,1)$ we must have

$$0 \le (\alpha+\delta) \le 2\,. \tag{27}$$

Using eqs. (23), (27) we see that (26), representing the maximum value of (24), must be negative or zero. As a result there is no cancellation between the two terms in the left hand side of (20). The only possibility is that both terms may vanish simultaneously. Vanishing of the second term will require

$$\gamma = 0, \qquad \lambda = \delta\,, \tag{28}$$

while from (23), (25)-(27) we see that the vanishing of the first term would require

$$\alpha\delta - \beta\gamma = 1, \qquad \alpha+\delta = 2, \qquad \lambda = \frac{\alpha+\delta}{2}\,. \tag{29}$$

As a consequence of (28), (29) we get

$$\alpha = \delta = 1, \qquad \gamma = 0, \qquad \lambda = 1\,. \tag{30}$$

We have seen however that for $\gamma = 0$, (17), (18) may contain additional information beyond the ones which have been already discussed. In particular substituting (30) into (18) we get $\beta = 0$. This choice of $(\alpha,\beta,\gamma,\delta)$ corresponds to the trivial case where the final decay products have charges $(\vec{Q},\vec{P})$ and $(0,0)$.

Similar analysis shows that the decay of a quarter BPS dyon into three or more quarter or half BPS dyons occur on subspaces of codimension larger than one, since this would require aligning multiple six dimensional vectors along a plane and/or aligning multiple two dimensional vectors along a line. This completes our proof that the only possible codimension one subspaces of marginal stability arise from the decay of a quarter BPS dyon into a pair of half BPS dyons.

Before concluding this paper we would like to offer a physical explanation of why decay of a quarter BPS state into quarter BPS states requires more constraint than decay into half BPS states. This essentially arises from the fact that at a point of marginal stability the supersymmetries of decay products must align. Since half-BPS states have more supersymmetry than quarter BPS states, it is clearly easier to ensure that a pair of half BPS states have one common supersymmetry than ensuring that a pair of quarter BPS states have a common supersymmetry. Similar argument can be given for the decay of a quarter BPS state into three or more states.

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